The fundamental Group and Covering Spaces

Recall: A loop in a topological space X is a path that starts and end at the same point $x_0 \in X$.

We might call x_0 a base point.

Def: The fundamental group of X relative to the base point x_0, normally denoted by $\pi_1(X, x_0)$, is the set of path homotopy classes of loops baset at x_0, with the operation $*$.

Remark: $\pi_1(X, x_0)$ is sometimes called the first homotopy group of X. ("There is a more general subject called Homotopy theory").

Ex: $\pi_1(\mathbb{R}^n, x_0)$: the fundamental group of the euclidean n-space is the trivial group $\{[e_{x_0}]\}$.

- If f is a loop in $\mathbb{R}^n$, then the straight line homotopy is a path homotopy between f and and the constant path at x_0.

$$F(x,t) = (1-t) f(x) + t x_0.$$

- If X is a convex set of $\mathbb{R}^n$, the same apply.

  precisely the unit ball

$$B^n = \{x \mid x_1^2 + \dots + x_n^2 \leq 1\}.$$

  has trivial fundamental group.

Def: Let α be a path in X from x_0 to x_1. Define the map

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

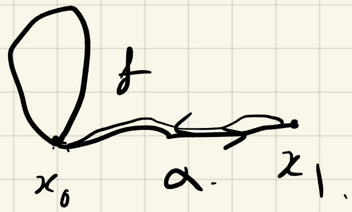
by the equation

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha].$$

The inverse.

if f is a loop at x_0.

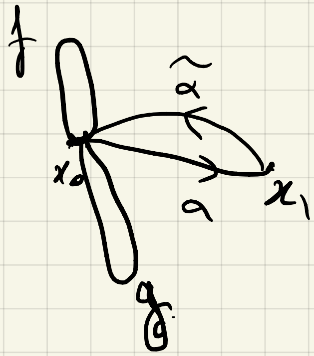
then $\bar{\alpha} * f * \alpha$. is a loop based at x_1



Theorem: The map $\hat{\alpha}$ is a group isomorphism.

proof: Step 1: $\hat{\alpha}$ is a homomorphism.

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = \left([\hat{\alpha}] * [f] * [\alpha]\right) * \left([\hat{\alpha}] * [g] * [\alpha]\right)$$



$$= [\hat{\alpha}] * [f] * [g] * [\alpha].$$

$$= \hat{\alpha}([f] * [g])$$

- We'll show that B that denotes $\bar{\alpha}$ is the inverse of $\hat{\alpha}$..

$\in \pi_1(X, x_1)$

$$\hat{\beta}([h]) = [\bar{\beta}] * [h] * [\beta] = [\alpha] * [h] * [\bar{\alpha}].$$

$$\hat{\alpha}(\hat{\beta}[h]) = [\bar{\alpha}] * ([\alpha] * [h] * [\bar{\alpha}]) * [\alpha].$$

$$= [h]$$

And Similarly $\hat{\beta}(\hat{\alpha}([f])) = [f]$ for all $[f] \in \pi_1(X, x_0)$

Corollary If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_1)$ is isomorphic to $\pi_1(X, x_1)$.

_ Deal only with path connected spaces when studying the fundamental group.

Def: X is simply connected if it is a path-connected space.

& $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$, consequently for every $x_0 \in X$. We denote this fact, $\pi_1(X, x_0)$ is trivial, by $\pi_1(X, x_0) = 0$.

Lemma: Suppose X is simply connected. Let f and g be two paths in X from x_0 to x_1, then $f \simeq_p g$.

Proof:

$f * \bar{g}$ is a loop on X based at x_0.

- this loop is path homotopic to a constant loop due to the fact That X is simply connected.

$$[\alpha * \bar{\beta}] * [\beta] = [e_{x_0}] * [\beta].$$

$$\Rightarrow [\alpha] = [\beta].$$

It seems that the fundamental group is a topological invariant, However, we want to prove it formally.

$\Rightarrow$ introduce homomorphism induced by a continuous map.

- Suppose $h : X \longrightarrow Y$ is a continous map.

  that carries the point $x_0 \in X$ to the point $y_0 \in Y$.

  Notation: $h : (X, x_0) \longrightarrow (Y, y_0)$.

  if f is a loop in X based at x_0, then $h \circ f : I \longrightarrow Y$ is a loop in Y based at y_0.

Def: Let $h : (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map. Define

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

by the equation

$$h_* : ([f]) = [h \circ f]$$

h_* is called homomorphism induced by h, relative to the base point x_0.

- h is homomorphism is due to

$$(h\circ f) * (h\circ g) = h\circ (f * g).$$

- $(h_{x_0})_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$.
- $(h_{x_1})_*: \pi_1(X, x_1) \longrightarrow \pi_1(Y, y_1)$.

...

If $x_0 = x$ then h_*.

Theorem (functorial properties)
If $h: (X, x_0) \longrightarrow (Y, y_0)$ and $k: (Y, y_0) \longrightarrow (Z, z_0)$ are continuous, then $(k\circ h)_* = k_* \circ h_*$.
If $i: (X, x_0) \longrightarrow (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Proof. By definition

$$(k\circ h)_*([f]) = [(k\circ h)\circ f].$$

$$(k_*\circ h_*)([f]) = k_*(h_*([f])) = k_*[h\circ f]$$
$$= [k\circ h\circ f].$$

Similarly, $i_*([f]) = [i\circ f] = [f]$.

Corollary: If $h: (X, x_0) \longrightarrow (Y, y_0)$ is a homeomorphism of X with Y, the h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(X, y_0)$.

Proof: Let $k : (Y, y_0) \longrightarrow (X, x_0)$ be the inverse of h. Then $k_* \circ h_* = (k \circ h)_* = i_*$, the identity map of (X, x_0). $h_* \circ k_* = (h \circ k)_* = j_*$, the identity map of (Y, y_0). Since i_* and j_* are the identity homomorphisms of the groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, k_* is the inverse of h_*.

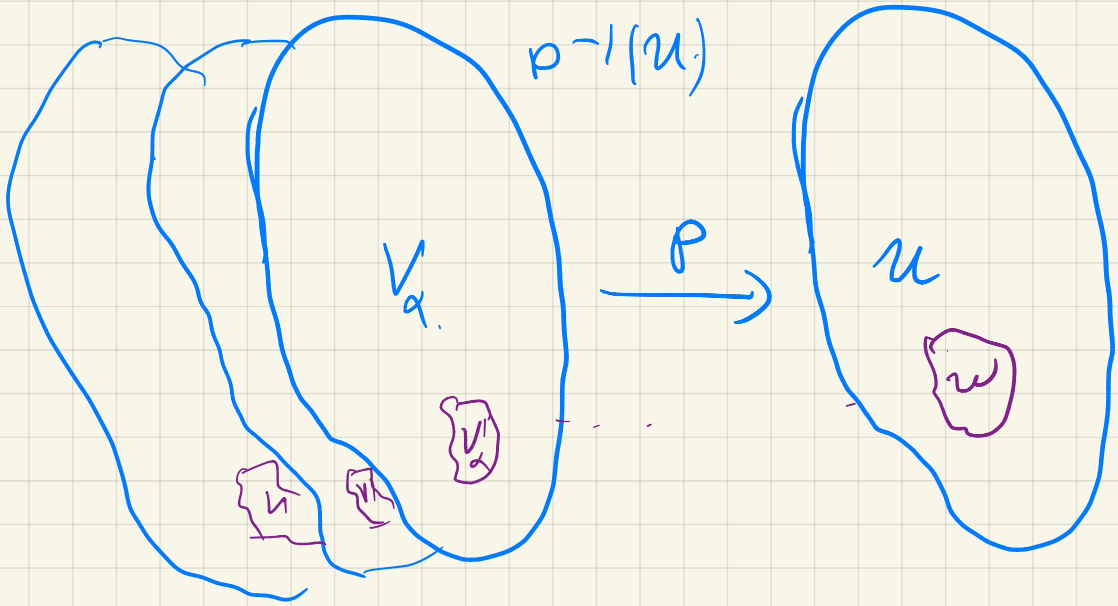
Covering Spaces.

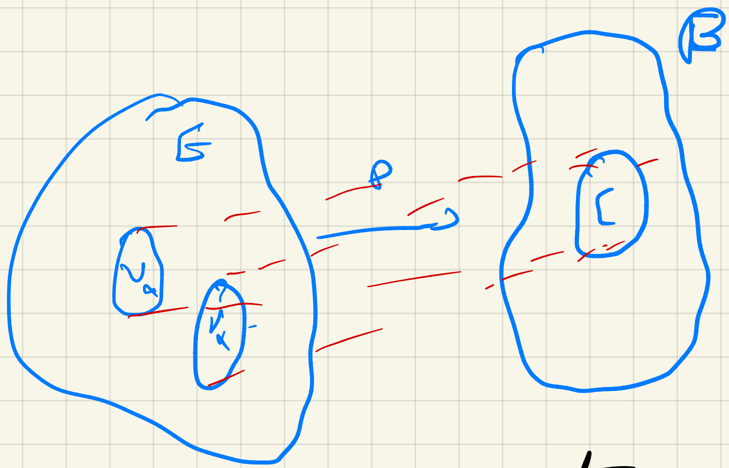
- Our goal is to compute some fundamental groups that are not trivial.
- The notion of covering space will be one of most important tools to carry this Task.
- Reimann Surfaces and complex manifolds. (May or may not Sudy them).

Def: Let $p : E \longrightarrow B$ be a continuous Surjective map.

- The open Set U of B is said to be *evenly covered* by p if the invese image of $p^{-1}(U)$ Can be written as the union of disjoint open Sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U.

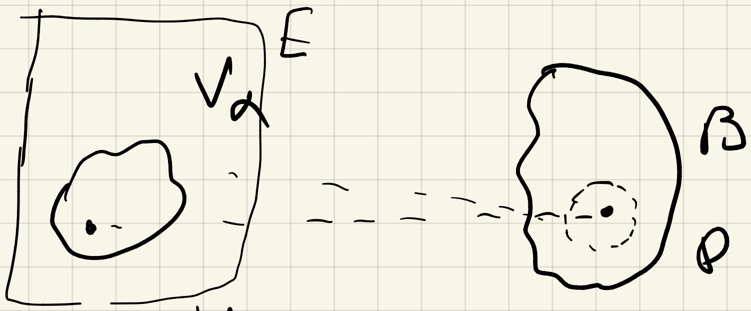
The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(\mathcal{U})$ into slices





Def: Let $p: E \longrightarrow B$ be a continous surjective. If every point p of B has a neighborhood $\mathcal{U}$ that is evenly covered by p, then p is called a covering map, and E is said to be a covering space of B.

Note 1 if $p: E \longrightarrow B$ is a covering map, then for each $b \in B$ the subspace $p^{-1}(b)$ of E has the discrete topology.



$p^{-1}(p) \cap V_\alpha$ is one point

therefore this point is open.

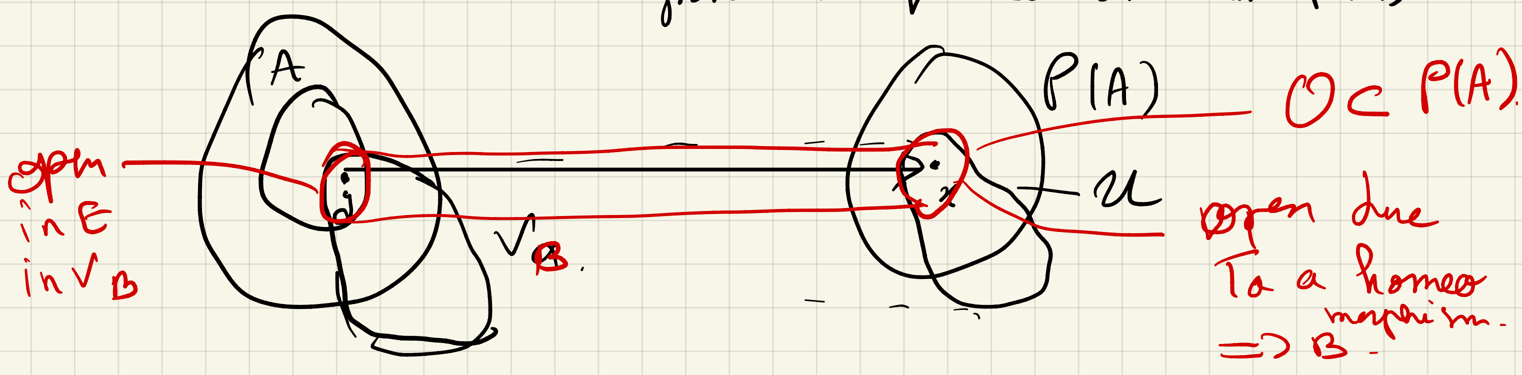
Note 2: if $p: E \longrightarrow B$ is a covering map, then p is an open map. that is it sends open set to open sets.

- Suppose A is open in E. Given $x \in p(A)$, choose a neighborhood $\mathcal{U}$ of x that is evenly covered by p.
  - $\{V_\alpha\}$ partition of $p^{-1}(\mathcal{U})$ into slices
  - There is $y \in A$ st $p(y) = x$.

    $y \in V_\beta$ then $V_\beta \cap A$ is open in E & open in V_β. p is homeomorphism onto $\mathcal{U}$

    $p(V_\beta \cap A)$ is open in $\mathcal{U}$ & hence open in B,

    It thus a neighborhood of x contained in $p(A)$.



Ex 1: Let X be any space; let $i : X \longrightarrow Y$ be identity map. Then i is a covering map (of the most trivial sort).

. Let $E = X \times \{1, 2, \ldots, n\}$: n disjoint copy of X.

Then the map $p(x, i) = x$ for all i

$E \longrightarrow X$

is also a trivial covering map.

Theorem: the map $p : \mathbb{R}^1 \longrightarrow S^1$ given by the equation

$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is a covering map.

proof. Next time.