The Fundemantal Group.

Main problem: Given two topological spaces, we want determine if they are homeomorphic or not.

What is homeomorphism?

A homeomorphism is a function $f: X \longrightarrow Y$ (X & Y are topological spaces) iff f is continuous bijection and $f^{-1}: Y \longrightarrow X$ is also continuous.

Equivalently: $f: X \longrightarrow Y$ is a bijection such that $f(\mathcal{U})$ is open iff $\mathcal{U}$ is open

Informaly: homeomorphic spaces have the same topological properties (connectedness, Compactness, local compactness, metrizability ...).

- To determine if two Topological spaces are homeomorphic, it is enough to construct a homeomorphism.

Otherwize: It is a different matter.

So what can we do? find a topological property that holds for one space but not for the other.

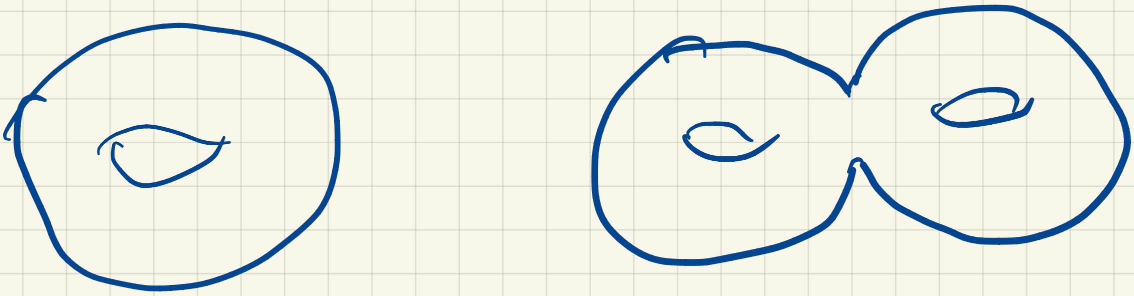
ex:

not homeomorphic

$$[0,1] \not\cong (0,1)$$

compact   not compact

For some spaces, the basic topological properties are not enough to show that they are not homeomorphic.



$\Rightarrow$ So, we will introduce, the Fundamental Group of a space.

"Two spaces are homeomorphic, then they have isomorphic Fundamental group."

Homotopy.

Given two continuous functions $f, g: X \to Y$ between two topological spaces, a homotopy from f to g is a continuous function

$$F: X \times I \longrightarrow Y$$

where $I = [0, 1]$ such that,

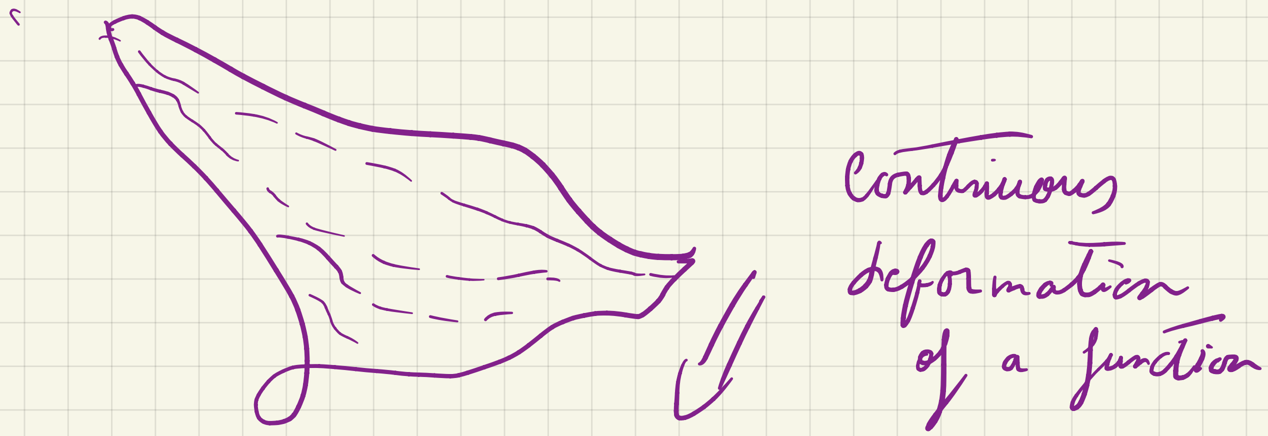
$$F(x, 0) = f(x)$$

& $F(x, 1) = g(x)$ for all x.

We say f and g are homotopic. We denote it as $f \simeq g$.

- If $f \simeq g$ & g is a constant map, then we say f is null homotopic.

- It is easy to see that $\simeq$ is equivalence relation. (easy exercise).



Property: The composition of two homotopic functions by two homotopic functions are homotopic. i.e

If $f, f' : X \longrightarrow Y$ & $g, g' : Y \longrightarrow Z$

& $f \simeq f'$ & $g \simeq g'$

then $g \circ f \simeq g' \circ f$.

Proof. Let $F : f \simeq f'$ & $G : g \simeq g'$

Define $H = G(H(x,t), t)$

for $t = 0$; $G(H(x,0), 0) = G(f(x), 0)$
$$= g(f(x)).$$

and for $t = 1$;

$$G(H(x,1),1) = G(f'(x), 1) = g'(f'(x)).$$

Thus, H is a homotopy from $g \circ f$ to $g' \circ f'$.

ex: Suppose $B \subset \mathbb{R}^n$ such that B is a convex set, and $f, g : X \longrightarrow B$ where X is a topological space (it is possible for $\mathcal{B}$ to be not convex but the segment conecting $f(x)$ & $g(x)$ must lies entirely in $\mathcal{B}$).

In this case we can define the following homotopy between the two functions.

$$H: X \times I \longrightarrow \mathcal{B}.$$

$$H(X, t) = (1-t) f(x) + t g(x)$$

we call H straight ligne homotopy between f & g.

(in this case all the functions to a convex set are homotopic).

Path Homotopy

Let X be a topological space,

a path in X is a continuous function $f:[0,1] \longrightarrow X$ such that

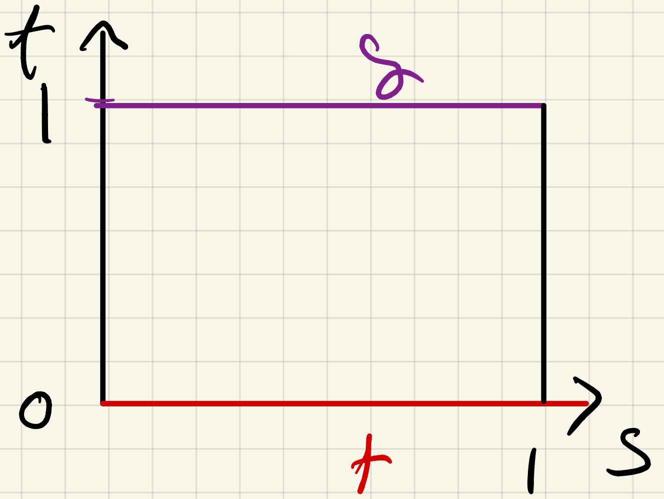
$$f(0) = x_0 \leftarrow \text{initial point}$$
$$f(1) = x_1 \leftarrow \text{final point.}$$

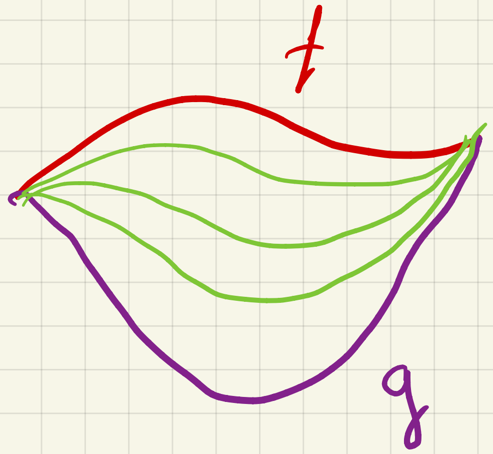
path homotopy, two paths $f, g: I \longrightarrow X$ are said to be path homotopic if they have the same initial point x_0 & the same final point x_1, and if there is a continuous function

$$F: I \times I \longrightarrow X \quad \text{st}$$

$$F(s,0) = f(s) \quad \& \quad F(0,t) = x_0$$
$$F(s,1) = g(s) \qquad F(1,t) = x_1$$

for $s, t \in I$.



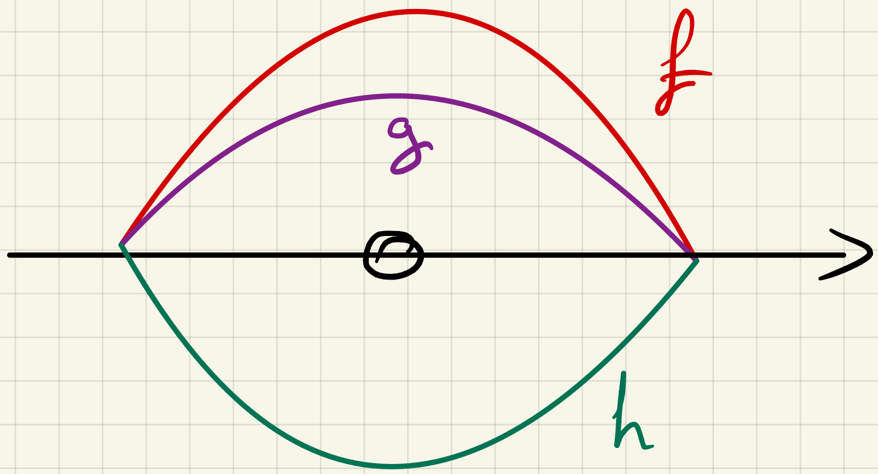


F is called path homotopy and f is path homotopic to g, denoted by $f \simeq_p g$.

Lemma: for any point $p, q \in X$, $\simeq_p$ is an equivalence relation on the set of all paths from p to q. (Exercise).

- If f is a path, we shall denote its path homotopy equivalence class by $[f]$.
- for the lemma, you need only the pasting lemma (gluing lemma).

ex:



$\mathbb{R}^2 - \{0\}$

$f \simeq_p g$

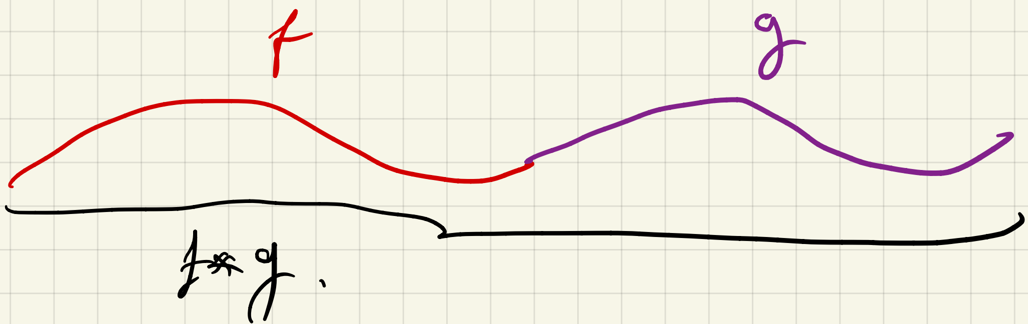
$f \not\simeq h$.

Now, we will define certain operation on the classes of path homotopy.

Def (path product): let $f, g : I \longrightarrow X$ be two paths such that $f(1) = f(0)$.

we will define $f * g$ as:

$$f \cdot g(s) = \begin{cases} f(2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$



Using this operation we can induce a well defined operation on path-homotopy classes

$$[f] * [g] = [f * g].$$

- Let e_x denotes the constant path.

$$e_x : I \longrightarrow X$$
$$e_x(t) = x \quad \text{for all } t.$$

- A path that starts and ends at the same point is called a loop.
- If f is a loop that starts & ends at $q \in X$ we say f is based at q. (q is the base point of f).

- $\Omega(X, q)$ will denote the set of all loops based at q. $e_x \in \Omega(X, q)$.

Properties of $(*)$

(1) associativity:

$$[f] * ([g] * [h]) = ([f] * [g]) * [h]$$

when $*$ is defined for the three paths.

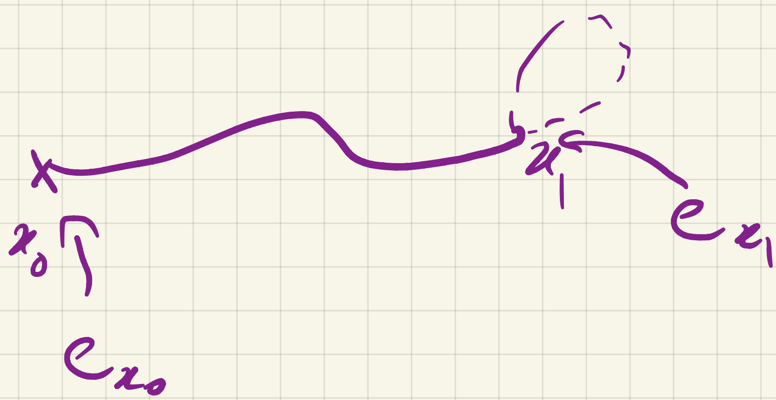
(2) right & left identities.

If f is a path from x_0 to x_1.

$$[f] * [e_{x_1}] = [f]$$

and $$[e_{x_0}] * [f] = [f].$$

ex:



(3) Inverse.

For a path f from x_0 to x_1

Define $\bar{f} : I \longrightarrow X$ to be the reverse of f $\quad \bar{f}(s) = f(1-s)$

so $[f] * [\bar{f}] = [e_{x_0}]$

and $\quad [\bar{f}] * [f] = [e_{x_1}]$.

Let $\pi_1(X, q)$ be the set of path classes of loops based at q.

Under the $(*)$ operation; $\pi_1(X, q)$ is a group called the fundamental group of X.